

# DYNAMIC AND GEOMETRIC PHASE FORMULAS IN THE HESTENES-DIRAC THEORY

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We examine the dynamic and geometric phases of the electron in quantum mechanics using Hestenes' spacetime algebra formalism. First the standard dynamic phase formula is translated into the spacetime algebra. We then define new formulas for the dynamic and geometric phases that can be used in Hestenes' formalism.

Key words: Dirac spinor, geometric algebra, geometric phase

## 1. INTRODUCTION

The geometric phase is now a standard topic in quantum mechanics [12]. To give an idea of what the geometric phase is, let us assume we have a parameter dependent Hamiltonian  $H(\mathbf{P}(t))$  where the parameters are time dependent [e.g., a time dependent magnetic field]. As a quantum system evolves under the Hamiltonian, it will pick up a total phase factor. We can decompose this phase into two parts. The first part is the dynamic phase which reflects how long the system has been evolving. The second part, the geometric phase part of the total phase, depends only on the path that  $\mathbf{P}(t)$  takes in its evolution. That is, the geometric phase reflects the geometry of  $\mathbf{P}$ 's evolution in parameter space. An important point is that the geometric phase does not depend on how fast  $\mathbf{P}(t)$  traverses its path, but depends only on the path itself. The article by Mukunda and Simon [11] offers perhaps the most easily understood comprehensive introduction to the theory.

Here we examine the dynamic and geometric phases in Hestenes' spacetime algebra formalism [6]. This is an alternative mathematical treatment of the Dirac equation. So, the system we have in mind is a relativistic spin-1/2 particle evolving under a time dependent Hamiltonian. The time dependence of the Hamiltonian will come from a time varying electromagnetic field.

The paper is organized as follows. In Section 2, we review the spacetime algebra formalism. The dynamic and geometric phase formulas are derived in Section 3. A discussion of the results follows in Section 4.

## 2. SPACETIME ALGEBRA

We review the spacetime algebra formalism that will be needed in the sequel. The reader is referred to [6] for a more detailed account. We will let  $\Psi$  and  $\Phi$  denote the traditional [i.e., complex] wavefunctions, and  $\psi$  and  $\phi$  denote their spacetime algebra representations.

Hestenes' spacetime algebra is the geometric [or Clifford] algebra of flat Minkowski spacetime. Before looking at this algebra, we will examine a simpler geometric algebra, the Pauli algebra. First, we will take as an axiom that the *vector* [or *Clifford*] *product* of a vector  $\mathbf{v}$  with itself is given by  $\mathbf{v}\mathbf{v} := \mathbf{v} \cdot \mathbf{v}$ , where  $\mathbf{v} \cdot \mathbf{v}$  is the usual *inner product*. For our example we will

have three vectors  $\sigma_1$ ,  $\sigma_2$  and  $\sigma_3$ . Let  $\mathbf{v} = \sigma_1 + \sigma_2$  so that

$$\begin{aligned}\mathbf{v}\mathbf{v} &= (\sigma_1 + \sigma_2)(\sigma_1 + \sigma_2) \\ &= \sigma_1\sigma_1 + \sigma_1\sigma_2 + \sigma_2\sigma_1 + \sigma_2\sigma_2\end{aligned}\tag{1}$$

and

$$\begin{aligned}\mathbf{v} \cdot \mathbf{v} &= (\sigma_1 + \sigma_2) \cdot (\sigma_1 + \sigma_2) \\ &= \sigma_1 \cdot \sigma_1 + 2\sigma_1 \cdot \sigma_2 + \sigma_2 \cdot \sigma_2.\end{aligned}\tag{2}$$

Since  $\mathbf{v}\mathbf{v} = \mathbf{v} \cdot \mathbf{v}$ , and using  $\sigma_1\sigma_1 = \sigma_1 \cdot \sigma_1$  and  $\sigma_2\sigma_2 = \sigma_2 \cdot \sigma_2$ , we see from (1) and (2) that

$$\sigma_1 \cdot \sigma_2 = \frac{1}{2}(\sigma_1\sigma_2 + \sigma_2\sigma_1).\tag{3}$$

Equation (3) allows us to define the inner product in terms of the vector product

$$\sigma_i \cdot \sigma_j = \frac{1}{2}(\sigma_i\sigma_j + \sigma_j\sigma_i).\tag{4}$$

Because  $\sigma_i \cdot \sigma_j$  is a scalar, we can form a metric tensor by defining  $\eta_{ij} := \sigma_i \cdot \sigma_j$ , where  $i, j = 1, 2, 3$ . Then (4) becomes

$$\sigma_i \cdot \sigma_j = \eta_{ij}.\tag{5}$$

Now notice that

$$\sigma_i\sigma_j = \frac{1}{2}(\sigma_i\sigma_j + \sigma_j\sigma_i) + \frac{1}{2}(\sigma_i\sigma_j - \sigma_j\sigma_i).\tag{6}$$

The first part of (6) is the inner product of  $\sigma_i$  and  $\sigma_j$ . The second part of (6) we will call the *outer product* of  $\sigma_i$  and  $\sigma_j$ , and denote this by  $\sigma_i \wedge \sigma_j := 1/2(\sigma_i\sigma_j - \sigma_j\sigma_i)$ . So (6) can be rewritten as

$$\sigma_i\sigma_j = \sigma_i \cdot \sigma_j + \sigma_i \wedge \sigma_j.\tag{7}$$

What happens if  $\sigma_i = \lambda\sigma_j$ , where  $\lambda$  is a scalar? Then,

$$\begin{aligned}\sigma_i \wedge \sigma_j &= \frac{1}{2}(\sigma_i\sigma_j - \sigma_j\sigma_i) \\ &= \frac{\lambda}{2}(\sigma_j\sigma_j - \sigma_j\sigma_j) \\ &= 0.\end{aligned}\tag{8}$$

So, if  $\sigma_i$  and  $\sigma_j$  are not a multiple of each other [i.e., if they are not collinear], our vector product will have a nonzero outer product.

The geometry of geometric algebra is seen by how we interpret all of this. As usual, we consider the inner product  $\sigma_i \cdot \sigma_j$  as being the projection of  $\sigma_i$  onto  $\sigma_j$ . But what of  $\sigma_i \wedge \sigma_j$ ? This we let represent the area between the vectors  $\sigma_i$  and  $\sigma_j$  in a plane that contains both vectors.

Let us now make some refinements. First, instead of representing a [real] scalar as simply  $\lambda$ , we will denote it by  $\lambda 1$ . Since scalars will commute with vectors [e.g.,  $\lambda \sigma_1 = \sigma_1 \lambda$ ], we will have

$$1\sigma_i = \sigma_i 1. \quad (9)$$

Now, what if our vectors are orthonormal so that

$$\sigma_i \cdot \sigma_j = \delta_{ij} \quad (10)$$

Then, if  $i \neq j$ ,

$$\begin{aligned} \sigma_i \sigma_j &= \sigma_i \cdot \sigma_j + \sigma_i \wedge \sigma_j \\ &= \sigma_i \wedge \sigma_j \\ &= -\sigma_j \wedge \sigma_i \\ &= -\sigma_j \sigma_i. \end{aligned} \quad (11)$$

So, if  $\sigma_i$  and  $\sigma_j$  are different orthonormal vectors, the outer product  $\sigma_i \wedge \sigma_j$  is not simply an area. Rather, it is a directed area since  $\sigma_i \sigma_j = -\sigma_j \sigma_i$ . So it depends on whether we go “clockwise” or “anti-clockwise” over the area. This is a generalization of the concept of a vector’s direction. To formalize this new concept, call the elements  $\sigma_i \sigma_j$  *bivectors*. Since  $i, j = 1, 2, 3$ , there are three independent bivectors given by  $\sigma_1 \sigma_2$ ,  $\sigma_1 \sigma_3$  and  $\sigma_2 \sigma_3$  [note that, for example,  $\sigma_1 \sigma_2 = -\sigma_2 \sigma_1$  and  $\sigma_3 \sigma_3 = \sigma_3 \cdot \sigma_3$ ]. A further generalization is to have a directed volume element, or a *trivector*,  $\mathcal{I} := \sigma_1 \sigma_2 \sigma_3$ . The trivector satisfies

$$\begin{aligned} \mathcal{I}^2 &= (\sigma_1 \sigma_2 \sigma_3)(\sigma_1 \sigma_2 \sigma_3) \\ &= -\sigma_1 \sigma_1 \sigma_2 \sigma_2 \sigma_3 \sigma_3 \\ &= -1. \end{aligned} \quad (12)$$

So, by starting with three vectors  $\sigma_1$ ,  $\sigma_2$  and  $\sigma_3$  and a vector product, we are naturally lead to new elements 1,  $\mathcal{I}$  and the bivectors. That is,

given three vectors, we can imagine, in addition to the vectors themselves, a scalar component, a directed volume and directed areas. So our basis for this construction is given by  $\{1, \sigma_i, \sigma_j \sigma_k, \sigma_1 \sigma_2 \sigma_3\}$ , where  $i = 1, 2, 3$  and  $1 \leq j < k \leq 3$ . Notice that

$$\begin{aligned}\mathcal{I}\sigma_1 &= \sigma_1 \sigma_2 \sigma_3 \sigma_1 \\ &= \sigma_1 \sigma_1 \sigma_2 \sigma_3 \\ &= \sigma_2 \sigma_3\end{aligned}\tag{13}$$

and that  $\mathcal{I}\sigma_2 = -\sigma_1 \sigma_3$  and  $\mathcal{I}\sigma_3 = \sigma_1 \sigma_2$ . So we can also use  $\{1, \sigma_i, \mathcal{I}\sigma_i, \mathcal{I}\}$ ,  $i = 1, 2, 3$ , as our basis. We prefer to use this latter basis.

It is possible to represent our vectors as matrices. For our example, the familiar Pauli spin matrices

$$\hat{\sigma}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \hat{\sigma}_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \hat{\sigma}_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\tag{14}$$

where  $i = \sqrt{-1}$ , are one possible representation. [In a matrix representation we let 1 be the identity matrix.] While it is nice to have an explicit matrix representation for our vectors, we must stress that they are to be treated as vectors. That means that under a spatial rotation, the basis vectors will change. For example, if we rotate in the  $\sigma_1 \sigma_2$ -plane by an angle  $\varphi$ , a vector will transform as

$$\mathbf{w} \rightarrow \mathbf{w}' = U \mathbf{w} U^{-1},\tag{15}$$

where  $U = \exp[-\mathcal{I}\sigma_3 \varphi/2]$ . So, if  $\varphi = \pi/2$ , we expect that  $\sigma_1 \rightarrow \sigma_2$ ,  $\sigma_2 \rightarrow -\sigma_1$  and  $\sigma_3 \rightarrow \sigma_3$ . This is easily checked since, e.g.,

$$\begin{aligned}\exp\left[-\mathcal{I}\sigma_3 \frac{\pi}{4}\right] \sigma_2 \exp\left[\mathcal{I}\sigma_3 \frac{\pi}{4}\right] &= \exp\left[-\mathcal{I}\sigma_3 \frac{\pi}{2}\right] \sigma_2 \\ &= \left[\cos\left(\frac{\pi}{2}\right) - \mathcal{I}\sigma_3 \sin\left(\frac{\pi}{2}\right)\right] \sigma_2 \\ &= -\mathcal{I}\sigma_3 \sigma_2 \\ &= -\sigma_1,\end{aligned}\tag{16}$$

where we used the facts that  $\mathcal{I}\sigma_i = \sigma_i \mathcal{I}$  in the first line, and that  $(\mathcal{I}\sigma_i)^2 = -1$  in the second. Equation (15) also holds for a general rotation using the Euler angles  $(\varphi, \theta, \chi)$ . In this case  $U$  has the form [4]

$$U = \exp\left[-\mathcal{I}\sigma_3 \frac{\varphi}{2}\right] \exp\left[-\mathcal{I}\sigma_2 \frac{\theta}{2}\right] \exp\left[-\mathcal{I}\sigma_3 \frac{\chi}{2}\right].\tag{17}$$

So far we have only considered scalars, vectors, etc. We can also have more general objects, called *multivectors*, which are of the form

$$C = c^s + (c_1^v \sigma_1 + c_2^v \sigma_2 + c_3^v \sigma_3) + \mathcal{I}(c_1^b \sigma_1 + c_2^b \sigma_2 + c_3^b \sigma_3) + c^t \mathcal{I}, \quad (18)$$

where  $C$  has scalar  $[c^s]$ , vector  $[c_i^v]$ , bivector  $[c_i^b]$  and trivector  $[c^t]$  parts. Let us introduce two operations on  $C$  that will prove useful in the sequel. The first is to let  $\langle C \rangle_s$  denote the scalar part of (18), so  $\langle C \rangle_s = c^s$ . For higher parts of  $C$ , we will let  $\langle C \rangle_i$  be the part of  $C$  that can be minimally expressed using  $i$  vectors. So,  $\langle C \rangle_1 = c_1^v \sigma_1 + c_2^v \sigma_2 + c_3^v \sigma_3$ ,  $\langle C \rangle_2 = \mathcal{I}(c_1^b \sigma_1 + c_2^b \sigma_2 + c_3^b \sigma_3)$  and  $\langle C \rangle_3 = c^t \mathcal{I}$ . The second operation is the *reversion* of  $C$ , written as  $\tilde{C}$ . By this we mean the reversal of all the vector products in  $C$ . So, for example,  $\widetilde{\sigma_1 \sigma_2} = \sigma_2 \sigma_1$ . In general, for two multivectors  $A$  and  $B$ , we have  $\widetilde{AB} = \tilde{B} \tilde{A}$ . For (18) notice that  $\tilde{\mathcal{I}} = -\mathcal{I}$  and  $\widetilde{\mathcal{I} \sigma_i} = -\sigma_i \mathcal{I} = -\mathcal{I} \sigma_i$  [because  $\mathcal{I} \sigma_i = \sigma_i \mathcal{I}$ ]. Then

$$\tilde{C} = c^s + (c_1^v \sigma_1 + c_2^v \sigma_2 + c_3^v \sigma_3) - \mathcal{I}(c_1^b \sigma_1 + c_2^b \sigma_2 + c_3^b \sigma_3) - c^t \mathcal{I}, \quad (19)$$

We can use the reversion operation to rewrite (15). From (17) we see that  $U^{-1} = \tilde{U}$ . So (15) can be written as

$$C \rightarrow C' = UC\tilde{U}, \quad (20)$$

where we now allow the rotation to act on a general multivector.

Now we will review Hestenes' spacetime algebra. The spacetime algebra is generated by four vectors  $\{\gamma_\mu\}$ ,  $\mu = 0, 1, 2, 3$ , that satisfy the Dirac algebra

$$\begin{aligned} \gamma_\mu \cdot \gamma_\nu &= g_{\mu\nu} \\ &= \text{diag}(+ \ - \ - \ -). \end{aligned} \quad (21)$$

The metric tensor  $g_{\mu\nu}$  in (21) is the usual one from special relativity. The  $\gamma_0$  vector is the time direction and the  $\gamma_n$ ,  $n = 1, 2, 3$ , vectors are the spatial directions, in an observer's frame of reference. We can also raise the indices on the  $\gamma_\mu$ 's by defining the vectors  $\gamma^\nu$  as those that satisfy the relation  $\gamma^\nu \cdot \gamma_\mu = \delta_\mu^\nu$ . [Notice that  $\gamma^0 = \gamma_0$  and  $\gamma^n = -\gamma_n$  satisfy this.] One possible matrix representation of the  $\gamma_\mu$ 's are the standard Dirac matrices.

The vectors  $\gamma_\mu$  result in the basis

$$\{1, \gamma_\mu, (\sigma_n, \mathcal{I} \sigma_n), \mathcal{I} \gamma_\mu, \mathcal{I}\}, \quad (22)$$

where  $n = 1, 2, 3$  and  $\sigma_n := \gamma_n \gamma_0$ . Also,  $\mathcal{I} := \gamma_0 \gamma_1 \gamma_2 \gamma_3 = \sigma_1 \sigma_2 \sigma_3$ ,  $\mathcal{I} \gamma_\mu = -\gamma_\mu \mathcal{I}$  and now  $\tilde{\mathcal{I}} = \mathcal{I}$ . The Pauli algebra we examined above is a sub-algebra of the Dirac algebra. So the 1,  $\sigma_n$ ,  $\mathcal{I} \sigma_n$  and  $\mathcal{I}$  in (22) satisfy the geometric algebra we looked at using the  $\sigma_n$ 's as our vectors. We can consider the Pauli algebra as the algebra of space and the Dirac algebra as the algebra of spacetime. Note that in the Pauli algebra, the  $\sigma_n$ 's are vectors while the  $\mathcal{I} \sigma_n$ 's are bivectors. In the Dirac algebra, both of these are bivectors.

In the spacetime algebra formalism, a proper, orthochronous Lorentz transformation [i.e., a spacetime rotation that does not reverse time or space] is represented by  $R$ . Since every Lorentz transformation can be decomposed into a pure boost and a pure spatial rotation,  $R$  will have the form  $R = LU$ , where  $U$  is the pure spatial rotation given in (17), and the pure boost,  $L$ , has the general form

$$L = \exp \left[ -\frac{(b_1 \sigma_1 + b_2 \sigma_2 + b_3 \sigma_3)}{2} \right]. \quad (23)$$

The spacetime generalization of (15) is given by

$$\mathbf{w} \rightarrow \mathbf{w}' = R \mathbf{w} R^{-1}. \quad (24)$$

We have already seen that  $U^{-1} = \tilde{U}$  which still holds in the Dirac algebra [because  $\tilde{\sigma}_n = -\sigma_n$ ,  $\tilde{\mathcal{I}} = \mathcal{I}$  and  $\mathcal{I} \sigma_n = \sigma_n \mathcal{I}$ , so  $(\tilde{\mathcal{I} \sigma_n}) = -\mathcal{I} \sigma_n$ ]. Because  $\tilde{\sigma}_n = -\sigma_n$ , we also have that  $L^{-1} = \tilde{L}$ , so  $\tilde{R} = \tilde{U} \tilde{L} = R^{-1}$ . As we did in (20), we can write (24) as

$$C \rightarrow C' = R C \tilde{R}, \quad (25)$$

where  $C$  is a general spacetime multivector.

It can be shown [8] that a Dirac spinor, in the spacetime algebra, has the general form

$$\psi = \left( \rho e^{\mathcal{I} \beta} \right)^{1/2} R, \quad (26)$$

where  $\rho(x)$  and  $\beta(x)$  are scalars and  $R(x)$  is a [proper, orthochronous] Lorentz transformation. [Deriving (26) is easy though lengthy, so we will simply accept this form for  $\psi$  without proof.] Note that  $\rho$ ,  $\beta$  and  $R$  depend on the spacetime point  $x$ . That is, they are locally defined parameters in the wavefunction  $\psi$ . In particular,  $R$  is a local Lorentz transformation—*not* a

global one. This observation leads to the local observables theory favored by Hestenes [7]. Now, the  $\rho$  term in (26) is interpreted as a proper probability density. A satisfactory interpretation of the  $\beta$  parameter is still lacking in Hestenes' theory, however. So Hestenes' interpretation of the Dirac equation, while attractive, is not yet complete. Using the form for  $R$  above, (26) can be rewritten as

$$\psi = (\rho e^{\mathcal{I}\beta})^{1/2} LU \quad (27)$$

$$= (\rho e^{\mathcal{I}\beta})^{1/2} LU_0 e^{-\mathcal{I}\sigma_3\chi/2}, \quad (28)$$

where, from (17),

$$U_0 := e^{-\mathcal{I}\sigma_3\varphi/2} e^{-\mathcal{I}\sigma_2\theta/2}. \quad (29)$$

Multiplying  $\Psi$  in the standard Dirac theory by a phase factor  $\exp[i\alpha]$  corresponds, in the Hestenes-Dirac theory, to

$$\psi \rightarrow \psi' = \psi e^{\mathcal{I}\sigma_3\alpha}. \quad (30)$$

Equation (30) simply expresses a correspondence between two different mathematical formalism, the standard Dirac theory and Hestenes' theory. It is useful to note two things though. First,  $\mathcal{I}\sigma_3$ , in Hestenes' formalism, corresponds to the complex number  $i$  in the standard Dirac theory. Also, given (28), multiplication by a phase factor in the standard Dirac theory corresponds to a change in the Euler angle  $\chi$  in the spacetime algebra formalism. This allows us to identify  $-\chi/2$  as the phase in Hestenes' theory. [Note the minus sign here. In (30) we *added* a new phase  $\alpha$  to the original phase. The minus sign follows from (28).] Another lengthy, but easy, derivation shows that the Hestenes form of the Dirac equation for a spin-1/2 particle of charge  $e$  and mass  $m$  is given by [with  $\hbar = 1$  and  $c = 1$ ]

$$\square\psi\mathcal{I}\sigma_3 - eA\psi = m\psi\gamma_0, \quad (31)$$

where  $A := \gamma^\mu A_\mu$  is the electromagnetic field vector and  $\square := \gamma^\mu \partial_\mu$ . [The Einstein summation convention is used here, so  $\gamma^\mu A_\mu := \gamma^0 A_0 + \gamma^1 A_1 + \gamma^2 A_2 + \gamma^3 A_3$ .]

In the standard Dirac theory, the probability density  $\varrho$  in an observer's frame of reference is associated with the time component of the Dirac current  $\Psi^\dagger\Psi$ , where  $\Psi^\dagger$  is the Hermitian adjoint of the complex wavefunction  $\Psi$ . In

order to translate this into Hestenes' formalism, we need to find the spacetime algebra equivalent of  $\Psi^\dagger\Phi$ , where  $\Phi$  is some other complex wavefunction. The spacetime algebra representation of  $\Psi^\dagger$  is given by [1]

$$\Psi^\dagger \leftrightarrow \psi^\dagger := \gamma_0 \tilde{\psi} \gamma_0. \quad (32)$$

Now,  $\Psi^\dagger\Phi = r + ic$  where  $r$  and  $c$  are real scalars. The real part of  $\Psi^\dagger\Phi$ , denoted  $\Re(\Psi^\dagger\Phi)$ , is simply the real scalar  $r$ . This corresponds to the scalar part of  $\phi\gamma_0\tilde{\psi}\gamma_0$ , the spacetime algebra representation of  $\Psi^\dagger\Phi$ . Then  $\Re(\Psi^\dagger\Phi) \leftrightarrow \langle\phi\gamma_0\tilde{\psi}\gamma_0\rangle_s$ . Now recall that  $i$  in the Dirac theory corresponds to  $\mathcal{I}\sigma_3$  in Hestenes' theory. Then the imaginary part of  $\Psi^\dagger\Phi$ , written as  $\Im(\Psi^\dagger\Phi) = -\Re(i\Psi^\dagger\Phi)$ , goes to  $-\langle\phi\mathcal{I}\sigma_3\gamma_0\tilde{\psi}\gamma_0\rangle_s$  in the spacetime algebra formalism. So the spacetime algebra representation of the complex probability amplitude density of standard quantum theory is given by [1]

$$\begin{aligned} \Psi^\dagger\Phi &= \Re(\Psi^\dagger\Phi) + i\Im(\Psi^\dagger\Phi) \\ &\leftrightarrow \langle\phi\gamma_0\tilde{\psi}\gamma_0\rangle_s - \mathcal{I}\sigma_3\langle\phi\mathcal{I}\sigma_3\gamma_0\tilde{\psi}\gamma_0\rangle_s. \end{aligned} \quad (33)$$

Equation (33) is equal to the probability density  $\varrho$  when  $\Phi = \Psi$ .

Notice the ubiquitous presence of  $\gamma_0$  in (32) and (33). Since  $\gamma_0$  is the time vector in an observer's frame, we say that the Hermitian adjoint  $\Psi^\dagger$  and the probability density  $\varrho$  are "frame dependent". That is, they depend on the direction of  $\gamma_0$ , and, hence, on the observer's frame of reference.

Results similar to (32) and (33) hold for the Dirac adjoint  $\bar{\Psi} := \Psi^\dagger\hat{\gamma}_0$ , where  $\hat{\gamma}_0$  is the standard Dirac matrix. The spacetime algebra representation of  $\bar{\Psi}$  is [1]

$$\bar{\Psi} \leftrightarrow \tilde{\psi}. \quad (34)$$

Also,

$$\bar{\Psi}\Phi \leftrightarrow \langle\phi\tilde{\psi}\rangle_s - \mathcal{I}\sigma_3\langle\phi\mathcal{I}\sigma_3\tilde{\psi}\rangle_s. \quad (35)$$

Since (34) and (35) do not contain a  $\gamma_0$  term, we say they are frame independent.

The  $\psi$  in (26) can also be rewritten in a form like that in (18). That is,  $\psi$  can be expressed as

$$\psi = c^s + (c_1^v\sigma_1 + c_2^v\sigma_2 + c_3^v\sigma_3) + \mathcal{I}(c_1^b\sigma_1 + c_2^b\sigma_2 + c_3^b\sigma_3) + c^t\mathcal{I}, \quad (36)$$

where  $\sigma_n = \gamma_n \gamma_0$  in the Dirac, versus Pauli, algebra. Now, all the  $c$ 's in (36) are real, so it would appear that we have managed to remove the complex structure of the Dirac theory by using the spacetime algebra. This is incorrect, as (33) shows. The “1” and “ $\mathcal{I}\sigma_3$ ” of Hestenes’ theory correspond to the real and imaginary parts, respectively, of the standard Dirac theory. So, even though we can express  $\psi$  completely in terms of real numbers, as in (36), the complex nature of the Dirac theory still remains. As explained in [3], such a complex structure is inherent to any correct formulation of Dirac’s equation.

Lastly, we need to define a few terms that will be used latter. [To rigorously justify these definitions will take us too far afield. The reader can consult [8], and the references therein, to find their physical justifications. For now, they can simply be taken as definitions.] The proper velocity vector  $v = v^\mu \gamma_\mu$  is defined by

$$v := R\gamma_0\tilde{R} \quad (37)$$

$$= R_0\gamma_0\tilde{R}_0, \quad (38)$$

where  $R$  is from the wavefunction  $\psi$  and  $R_0 := LU_0$ . That is,  $v$  is given by the spacetime rotation of the time axis  $\gamma_0$  determined locally by the wavefunction  $\psi$ . We have that  $vv = 1$ , so the velocity vector is normalized. From (33) we can see that, when  $\Phi = \Psi$ ,

$$\varrho = \rho v^0, \quad (39)$$

where  $\rho$  is also from the wavefunction  $\psi$ . We define the angular velocities as

$$\Omega_\mu := 2(\partial_\mu R)\tilde{R} \quad (40)$$

$$\omega_\mu := 2(\partial_\mu R_0)\tilde{R}_0. \quad (41)$$

Because  $R\tilde{R} = 1$ , we have that  $(\partial_\mu R)\tilde{R} = -R(\partial_\mu \tilde{R})$ . Notice that this implies that  $\Omega_\mu = -\tilde{\Omega}_\mu$ . Similarly, we can show that  $\omega_\mu = -\tilde{\omega}_\mu$ . Now, the spin polarization vector  $s = s^\mu \gamma_\mu$  is given by

$$s := \frac{1}{2}R\gamma_3\tilde{R} \quad (42)$$

$$= \frac{1}{2}R_0\gamma_3\tilde{R}_0. \quad (43)$$

Finally, the spin angular momentum bivector  $S = \mathcal{I}sv$  is defined by

$$S := \frac{1}{2}R\mathcal{I}\sigma_3\tilde{R} \quad (44)$$

$$= \frac{1}{2}R_0\mathcal{I}\sigma_3\tilde{R}_0. \quad (45)$$

Equation (44) gives us  $\partial_\mu S = 1/2(\Omega_\mu S - S\Omega_\mu)$ . Similarly, from (45), we have  $\partial_\mu S = 1/2(\omega_\mu S - S\omega_\mu)$ . From (38), (43) and (45) we see that  $R_0$  determines the orientations of  $v$ ,  $s$  and  $S$ . By orientation we mean the directions of the vectors  $v$  and  $s$ , and the “tilt” and direction of the directed area  $S$ , in the four dimensional spacetime.

### 3. PHASE FORMULAS

Here we derive the spacetime algebra formulas for the dynamic and geometric phases. Initially we simply translate the standard dynamic phase formula into Hestenes’ formalism. But, much of Hestenes’ theory deals with local observables [7], so we will define a local dynamic phase. This local definition is useful for working in the spacetime algebra. Next we derive the geometric phase formula. We then briefly examine the relationships between the spacetime algebra and standard formulas. Finally, we will find that we are able to redefine the phases, allowing us to use much simpler formulas in Hestenes’ theory.

First, notice that (31) is invariant under the transformation

$$\psi \rightarrow \psi \exp[\mathcal{I}\sigma_3\alpha], \quad (46)$$

where  $\alpha$  is a real constant. In the Dirac theory, this corresponds to adding a constant phase to  $\Psi$ . In Hestenes’ theory, we are adding a constant angle to the Euler angle  $\chi$  in (28). Let us now define the projection operator  $\Pi$  by

$$\Pi(\psi) := \left\{ \psi' : \psi' = \psi e^{\mathcal{I}\sigma_3\alpha}, \text{ for all real constants } \alpha \right\} \quad (47)$$

and let  $\mathcal{P}$  be the space of all such projections. This amounts to projecting the multivectors  $\psi$  in the Hilbert space  $\mathcal{H}$  onto the representative ray  $\Pi(\psi)$  in  $\mathcal{P}$ , where  $\Pi(\psi)$  differs from  $\psi$  only by a constant spatial rotation  $\exp[\mathcal{I}\sigma_3\alpha]$ . [Equation (47) corresponds to the projection of  $\Psi$  onto the ray  $\hat{\Pi}(\Psi)$ , where  $\hat{\Pi}(\Psi)$  differs from  $\Psi$  only by a phase factor  $\exp[i\alpha]$ .] If  $\psi(\xi)$  evolves along the

curve  $\mathcal{C}$  in  $\mathcal{H}$ , then  $\Pi(\psi(\xi))$  will evolve along the curve  $\hat{\mathcal{C}}$  in  $\mathcal{P}$ . Notice that we have parameterized  $\psi$ 's evolution in  $\mathcal{H}$  by  $\xi$ . The geometric phase should only depend on the curve  $\hat{\mathcal{C}}$  in  $\mathcal{P}$ , the path of  $\Pi(\psi)$  in  $\mathcal{P}$ . In particular, the geometric phase needs to be independent of the rate at which  $\Pi(\psi)$  traverses  $\hat{\mathcal{C}}$ . Therefore, a reparameterization of  $\xi$  can not affect the geometric phase.

The [complex] global dynamic phase,  $\Delta_G$ , is given by [11]

$$\Delta_G = \int d\xi d^3x \Im(\Psi^\dagger \dot{\Psi}), \quad (48)$$

where the overdot represents differentiation with respect to  $\xi$ . As we have seen earlier,  $\Im(\Psi^\dagger \dot{\Psi}) \leftrightarrow -\langle \dot{\psi} \mathcal{I} \sigma_3 \gamma_0 \tilde{\psi} \gamma_0 \rangle_s$ . So the spacetime algebra formula for the global dynamic phase,  $\delta_G$ , is

$$\delta_G = - \int d\xi d^3x \langle \dot{\psi} \mathcal{I} \sigma_3 \gamma_0 \tilde{\psi} \gamma_0 \rangle_s. \quad (49)$$

From (49) let us define the local dynamic phase by

$$\varrho \dot{\delta}_L := -\langle \dot{\psi} \mathcal{I} \sigma_3 \gamma_0 \tilde{\psi} \gamma_0 \rangle_s. \quad (50)$$

We use  $\varrho$ , rather than  $\rho$ , in (50) because the Hermitian adjoint used in (49) is frame dependent, so we expect the same for our probability distribution. Also, since the Hermitian adjoint singles out a preferred time direction, we will let  $\xi$  be the *observer's time*, given by  $t$ . Using (26) in (50) results in, after some tedious algebra,

$$\begin{aligned} \varrho \dot{\delta}_L &= \rho \dot{\beta} \langle \frac{1}{2} R \gamma_3 \tilde{R} \gamma_0 \rangle_s - \rho \langle \dot{R} \mathcal{I} \gamma_3 \tilde{R} \gamma_0 \rangle_s \\ &= \rho \dot{\beta} \langle s \gamma_0 \rangle_s - \rho \langle \Omega_0 \mathcal{I} s \gamma_0 \rangle_s \\ &= \rho s^0 \dot{\beta} - \rho \langle \Omega_0 S v \gamma_0 \rangle_s \\ &= \varrho \frac{s^0}{v^0} \dot{\beta} - \rho \langle \Omega_0 S (v^0 + \mathbf{v}) \rangle_s, \end{aligned} \quad (51)$$

where  $v \gamma_0 := v^0 + \mathbf{v}$ ,  $\mathbf{v} := v^n \sigma_n$ , and  $Sv = \mathcal{I}s$  because  $vv = 1$ . The  $\Omega_0 S$  term in (51) can have scalar, bivector and pseudoscalar [i.e.,  $\mathcal{I}$ ] parts, denoted by  $\Omega_0 \cdot S$ ,  $\langle \Omega_0 S \rangle_2$  and  $\langle \Omega_0 S \rangle_4$ , respectively. Under the reversion operation, the scalar and pseudoscalar parts are even, while the bivector part is odd. Hence,

$$\begin{aligned} \langle \Omega_0 S \rangle_2 &= \frac{1}{2} (\Omega_0 S - (\widetilde{\Omega_0 S})) \\ &= \frac{1}{2} (\Omega_0 S - S \Omega_0). \end{aligned} \quad (52)$$

From (44), we see that

$$\begin{aligned}\dot{S} &= \frac{1}{2}(\dot{R}\mathcal{I}\sigma_3\tilde{R} + R\mathcal{I}\sigma_3\dot{\tilde{R}}) \\ &= \langle \Omega_0 S \rangle_2.\end{aligned}\tag{53}$$

Using this in (51) gives us

$$\begin{aligned}\varrho \dot{\delta}_L &= \varrho \frac{s^0}{v^0} \dot{\beta} - \rho \langle (\Omega_0 \cdot S + \dot{S} + \langle \Omega_0 S \rangle_4)(v^0 + \mathbf{v}) \rangle_s \\ &= \varrho \frac{s^0}{v^0} \dot{\beta} - \varrho \Omega_0 \cdot S - \rho \langle \dot{S} \mathbf{v} \rangle_s.\end{aligned}\tag{54}$$

Now,  $s \cdot v = 0$ , so

$$s^0 = \frac{\mathbf{s} \cdot \mathbf{v}}{v^0},\tag{55}$$

where  $s\gamma_0 = s^0 + \mathbf{s}$ . Thus, (54) becomes

$$\dot{\delta}_L = -\Omega_0 \cdot S - \frac{\mathbf{v}}{v^0} \cdot \left[ \dot{S} - \frac{\mathbf{s}}{v^0} \dot{\beta} \right].\tag{56}$$

From (49) to (56), it follows that, after letting  $\xi = t$ ,

$$\begin{aligned}\delta_G &= \int dt d^3x \varrho \dot{\delta}_L \\ &= \int dt d^3x \varrho \left\{ -\Omega_0 \cdot S - \frac{\mathbf{v}}{v^0} \cdot \left[ \dot{S} - \frac{\mathbf{s}}{v^0} \dot{\beta} \right] \right\}.\end{aligned}\tag{57}$$

We now find a spacetime algebra geometric phase formula. First, let us define a new wavefunction  $\psi' := \psi \exp[-\mathcal{I}\sigma_3\delta_L]$  that differs from  $\psi$  only in having the local dynamic phase removed. Now,

$$\dot{\psi}' = \dot{\psi} e^{-\mathcal{I}\sigma_3\delta_L} + \psi e^{-\mathcal{I}\sigma_3\delta_L} (-\mathcal{I}\sigma_3\dot{\delta}_L).\tag{58}$$

Using (33) for the inner product of  $\dot{\psi}'$  with  $\psi'$ , and noting that  $\langle \psi \mathcal{I}\sigma_3\gamma_0 \tilde{\psi} \gamma_0 \rangle_s = 0$ , we have from (58)

$$\begin{aligned}\langle \dot{\psi}' \gamma_0 \tilde{\psi}' \gamma_0 \rangle_s - \mathcal{I}\sigma_3 \langle \dot{\psi}' \mathcal{I}\sigma_3 \gamma_0 \tilde{\psi}' \gamma_0 \rangle_s &= \langle \dot{\psi} \gamma_0 \tilde{\psi} \gamma_0 \rangle_s - \mathcal{I}\sigma_3 \langle \dot{\psi} \mathcal{I}\sigma_3 \gamma_0 \tilde{\psi} \gamma_0 \rangle_s \\ &\quad - \mathcal{I}\sigma_3 \dot{\delta}_L \langle \psi \gamma_0 \tilde{\psi} \gamma_0 \rangle_s \\ &= \langle \dot{\psi} \gamma_0 \tilde{\psi} \gamma_0 \rangle_s + \mathcal{I}\sigma_3 \varrho \dot{\delta}_L - \mathcal{I}\sigma_3 \varrho \dot{\delta}_L \\ &= \langle \dot{\psi} \gamma_0 \tilde{\psi} \gamma_0 \rangle_s.\end{aligned}\tag{59}$$

Since  $\langle \dot{\psi}' \gamma_0 \tilde{\psi}' \gamma_0 \rangle_s = \langle \dot{\psi} \gamma_0 \tilde{\psi} \gamma_0 \rangle_s$ , it follows from (59) that

$$\langle \dot{\psi}' \mathcal{I} \sigma_3 \gamma_0 \tilde{\psi}' \gamma_0 \rangle_s = 0. \quad (60)$$

With  $\psi'$  differing from  $\psi$  only by the  $\chi$  factor of (28), let us write

$$\psi' = \left( \rho e^{\mathcal{I} \beta} \right)^{1/2} L U_0 e^{-\mathcal{I} \sigma_3 \chi' / 2}. \quad (61)$$

Using (61) in (60), and doing some lengthy algebra, gives us, similar to our derivation of (54),

$$\begin{aligned} \langle \dot{\psi}' \mathcal{I} \sigma_3 \gamma_0 \tilde{\psi}' \gamma_0 \rangle_s &= \frac{\rho}{2} \langle (-R_0 \mathcal{I} \sigma_3 \dot{\chi}' + \dot{\beta} \mathcal{I} R_0 + 2 \dot{R}_0) \mathcal{I} \sigma_3 \gamma_0 \tilde{R}_0 \gamma_0 \rangle_s \\ &= \frac{\rho}{2} \dot{\chi}' - \rho \frac{s^0}{v^0} \dot{\beta} + \rho \langle \omega_0 S (v^0 + \mathbf{v}) \rangle_s \\ &= \frac{\rho}{2} \dot{\chi}' - \rho \frac{s^0}{v^0} \dot{\beta} + \rho \omega_0 \cdot S + \rho \langle \dot{S} \mathbf{v} \rangle_s \\ &= \frac{\rho}{2} \dot{\chi}' - \rho \frac{s^0}{v^0} \dot{\beta} + \rho \omega_0 \cdot S + \rho \frac{\mathbf{v}}{v^0} \cdot \dot{S}, \end{aligned} \quad (62)$$

where we used the fact that  $\dot{S} = \langle \omega_0 S \rangle_2$  [see (45)]. Then, from (60) and (62), we have

$$\frac{1}{2} \dot{\chi}' = -\omega_0 \cdot S - \frac{\mathbf{v}}{v^0} \cdot \left[ \dot{S} - \frac{\mathbf{s}}{v^0} \dot{\beta} \right]. \quad (63)$$

Notice that all of the terms on the righthand-side of (63) depend only on the path  $\hat{\mathcal{C}}$  in  $\mathcal{P}$ . That is, they are determined by  $R_0$ , not by  $R$ . [This does not hold for  $\delta_L$  because of the presence of  $\Omega_0$  in (56).] Also, both sides of (63) are linear in the time derivative. Hence,  $\chi'$  is independent of a reparameterization of  $t$ . These two observations allow us to define the local spacetime algebra geometric phase by  $\gamma_L := -\chi'/2 + \chi(0)/2$ . [We use a minus sign here because  $-\chi'/2 = -\chi/2 - \delta_L$ . The  $\chi(0)/2$  factor is because  $\gamma_L(0)$  must vanish. Then  $\gamma_L + \delta_L = -\chi/2 + \chi(0)/2$ , where we have previously identified  $-\chi/2$  as the total phase.] Thus,

$$\dot{\delta}_L = -\Omega_0 \cdot S - \frac{\mathbf{v}}{v^0} \cdot \left[ \dot{S} - \frac{\mathbf{s}}{v^0} \dot{\beta} \right] \quad (64)$$

$$\dot{\gamma}_L = \omega_0 \cdot S + \frac{\mathbf{v}}{v^0} \cdot \left[ \dot{S} - \frac{\mathbf{s}}{v^0} \dot{\beta} \right], \quad (65)$$

where  $\delta_L(0) = 0$  and  $\gamma_L(0) = 0$ . To find the global spacetime algebra geometric phase,  $\gamma_G$ , we need to perform the integration

$$\begin{aligned}\gamma_G &:= \int dt d^3x \varrho \dot{\gamma}_L \\ &= \int dt d^3x \varrho \left\{ \omega_0 \cdot S + \frac{\mathbf{v}}{v^0} \cdot \left[ \dot{S} - \frac{\mathbf{s}}{v^0} \dot{\beta} \right] \right\},\end{aligned}\quad (66)$$

similar to (57) above.

Equation (64) is a straightforward translation of the standard dynamic phase formula into the spacetime algebra. Thus, the value of  $\delta_G$  will equal that of  $\Delta_G$ . We may ask if the same holds true for  $\gamma_G$  and the standard geometric phase  $\Gamma_G$ ? This will not generally be the case. The  $-\chi/2$  factor in (28), which we identified as the total phase, may contain dynamics beyond the local representations of  $\Delta_G$  and  $\Gamma_G$ . As an example, let us consider an adiabatic evolution of a wavefunction that starts in an energy eigenstate. Let the  $m$ -th energy eigenfunction be given by

$$\phi_m = \left( \rho e^{\mathcal{I}\beta} \right)^{1/2} R_{m0} e^{-\mathcal{I}\sigma_3 \chi_m / 2}. \quad (67)$$

The adiabatic theorem states that our wavefunction  $\psi_m$  can differ from  $\phi_m$  only by a phase factor. Thus, using the complex global phases  $\Delta_{mG}$  and  $\Gamma_{mG}$

$$\psi_m = \phi_m e^{\mathcal{I}\sigma_3 (\Delta_{mG} + \Gamma_{mG})}. \quad (68)$$

[Strictly speaking, (68) will contain some  $\mathcal{O}(\varepsilon)$  terms where  $\varepsilon$  is an infinitesimal number that reflects the degree of adiabaticity, and the  $\mathcal{O}(\varepsilon)$  terms are due to departures from strict adiabaticity. We will ignore these terms in (68).] In this situation,  $\Gamma_{mG}$  is given by [13]

$$\dot{\Gamma}_{mG} = - \int d^3x \Im \left( \Phi_m^\dagger \dot{\Phi}_m \right). \quad (69)$$

A derivation similar to that in Section 3 for the dynamic phase shows that

$$\begin{aligned}\dot{\Gamma}_{mG} &= \int d^3x \left( \dot{\gamma}_{mL} + \frac{\dot{\chi}}{2} \right) \\ &= \dot{\gamma}_{mG} + \int d^3x \frac{\dot{\chi}}{2}.\end{aligned}\quad (70)$$

This result does not invalidate calling  $\gamma_L$  the spacetime algebra geometric phase. It still represents that part of the total phase  $-\chi/2$  due only to the geometry of the wavefunction's evolution.

Now, (26) can be rewritten as

$$\begin{aligned}\psi &= (\rho e^{\mathcal{I}\beta})^{1/2} R_0 e^{\mathcal{I}\sigma_3(\gamma_L + \delta_L - \chi(0)/2)} \\ &= (\rho e^{\mathcal{I}\beta})^{1/2} R_0 e^{\mathcal{I}\sigma_3(\int^t d\tau(\omega_0 \cdot S - \Omega_0 \cdot S) - \chi(0)/2)}.\end{aligned}\quad (71)$$

Thus, we can redefine the phases as

$$\dot{\hat{\delta}}_L := -\Omega_0 \cdot S \quad (72)$$

$$\dot{\hat{\gamma}}_L := \omega_0 \cdot S, \quad (73)$$

where  $\hat{\delta}_L(0) = 0$  and  $\hat{\gamma}_L(0) = 0$ . Equations (72) and (73) are still physically meaningful definitions. They capture all the essential properties of the dynamic and geometric phases. The simplicity of (72) and (73), versus (64) and (65), respectively, suggests that they are more useful definitions for Hestenes' formalism. It is easy to show, for example, that (72) and (73) hold exactly in the non-relativistic limit. That is, the second terms on the right-hand sides of (64) and (65) vanish in the non-relativistic limit because  $\mathbf{v} \rightarrow \mathbf{0}$ . Thus, we can use exactly the same phase formulas in the relativistic and non-relativistic cases. Also, it is easy to show the (64) and (65) reduce to (72) and (73), respectively, when we consider the adiabatic evolution of an energy eigenstate. Overall, (72) and (73) seem to be more reasonable definitions of the phases, in the spacetime algebra, than (64) and (65). Finally, we note that Hestenes' had previously proposed using  $\Omega_0 \cdot S$  as the total local phase formula [10]. As we see, except for the sign, this is only the dynamic part of the phase. We also need to take into account the geometric part of the total local phase.

The formulas (72) and (73) can be written as

$$\dot{\hat{\delta}}_L = -\langle \dot{R} \mathcal{I} \sigma_3 \tilde{R} \rangle_s \quad (74)$$

$$\dot{\hat{\gamma}}_L = \langle \dot{R}_0 \mathcal{I} \sigma_3 \tilde{R}_0 \rangle_s. \quad (75)$$

Unlike (32) and (33), these no longer have  $\gamma_0$  present. Thus, they are frame independent quantities. Now, the wavefunction  $\psi$  determines a set of *streamlines* via the proper velocity vector  $v$ . That is, if we imagine a particle as actually starting at some initial spacetime point  $x_0$ , the spacetime vector  $v$  will determine its future positions, given by the streamlines. Along a given streamline there is a proper time  $\tau$ . Because (74) and (75) are independent

of a particular reference frame, we can allow the derivative to be with respect to  $\tau$ . Hence, (74) and (75) can also be used for the proper phase formulas [remembering that, in this case, we are using a proper time derivative].

#### 4. DISCUSSION

Let us first review a few facts about  $\beta$  that will be useful in the following discussion [see the article by Gull, Lasenby and Doran [5] for more background]. When there is no electromagnetic field, i.e.,  $A = 0$  in (31), the Dirac equation admits plane wave solutions. For the electron solutions  $\beta = 0$ , while for the positron solutions  $\beta = \pi$ . However, when  $A \neq 0$ , a general wavefunction can have other values of  $\beta$ , as demonstrated by the [non-relativistic] solutions for the hydrogen atom. Additionally, in their numerical simulations of tunnelling times, Gull et al. show that  $\rho$  and  $\beta$  are not necessarily constant at a given position for all time. So the  $\dot{\beta}$  term in (64) does not necessarily vanish.

Now, it can be shown that [1]

$$\hat{\gamma}_5 \Psi \leftrightarrow \psi \sigma_3 \quad (76)$$

$$\bar{\Psi} \Psi \leftrightarrow \rho \cos(\beta) \quad (77)$$

$$\bar{\Psi} i \hat{\gamma}_5 \Psi \leftrightarrow -\rho \sin(\beta), \quad (78)$$

where  $\hat{\gamma}_5$  is the standard Dirac matrix. It follows that

$$\begin{aligned} \bar{\Psi} \Psi - i \hat{\gamma}_5 \bar{\Psi} i \hat{\gamma}_5 \Psi &= \rho e^{i \hat{\gamma}_5 \beta} \\ &\leftrightarrow \rho e^{\mathcal{I} \beta}. \end{aligned} \quad (79)$$

So the  $e^{\mathcal{I} \beta}$  term can be thought of as a local *chiral transformation* [2], an observation made previously [9]. Let us take this literally and think of  $\beta$  as the *chiral angle* in the Dirac wavefunction.

Equation (64) for the dynamic phase is a straightforward translation of the standard phase formula (48) into the geometric algebra. We can rewrite (48) as

$$\begin{aligned} \dot{\Delta}_G &= \int d^3x \Im(\Psi^\dagger \dot{\Psi}) \\ &= - \int d^3x \Psi^\dagger i \frac{d}{dt} \Psi. \end{aligned} \quad (80)$$

In the standard Dirac theory, (80) is interpreted as the negative of the expected value of the energy operator. The same physical interpretation is then given to (64), locally. However, interpreting (64) using only the spacetime algebra formalism is difficult because of the presence of  $\beta$ .

In contrast, the dynamic phase formula in (72) is easily interpreted in the spacetime algebra. It is the negative of the component of  $\Omega_0$  in the spacetime plane  $S$ . To translate (72) back into the standard Dirac theory, we use (74) and (35). If  $R$  is the spacetime representation of  $\mathbf{R}$ , in the standard theory (74) is given by

$$\dot{\hat{\Delta}}_L = \Im(\overline{\mathbf{R}}\dot{\mathbf{R}}). \quad (81)$$

Because  $\tilde{R}R = 1$ , we have from (79) that  $\overline{\mathbf{R}}\mathbf{R} = 1$ . It follows that  $\overline{\mathbf{R}}\dot{\mathbf{R}} = -(\overline{\mathbf{R}}\dot{\mathbf{R}})^\dagger$ . Hence,  $\overline{\mathbf{R}}\dot{\mathbf{R}}$  is purely imaginary. So (81) becomes

$$\dot{\hat{\Delta}}_L = -i\overline{\mathbf{R}}\dot{\mathbf{R}}. \quad (82)$$

The question now is, how to find  $\mathbf{R}$ ? Since

$$i\hat{\gamma}_5\Psi \leftrightarrow \mathcal{I}\psi \quad (83)$$

we can write (26) in the standard theory as

$$\Psi = \sqrt{\rho}e^{i\hat{\gamma}_5\beta/2}\mathbf{R}. \quad (84)$$

Provided  $\rho \neq 0$ , we have that

$$\mathbf{R} = \frac{e^{-i\hat{\gamma}_5\beta/2}}{\sqrt{\rho}}\Psi. \quad (85)$$

So  $\mathbf{R}$  is given by a local chiral transformation of  $\Psi$ . Notice that (84) implies that  $\mathbf{R}$  is invariant under a local chiral transformation of  $\Psi$ .

For both sets of phase formulas, we see that they are easily interpreted in one formalism but not the other. Both geometric phase definitions are invariant under a local gauge transformation. [This likely accounts for the difference in  $\gamma_G$  and  $\Gamma_G$ , see (70).] The phases in (72) and (73) are also invariant under a local chiral transformation. These may prove to be more useful in the electroweak theory.

It is difficult to decide theoretically which set of formulas is the correct one. It may also be difficult to experimentally verify which set is correct.

This is because (64) and (65) reduce to (72) and (73) for adiabatic evolutions of energy eigenstates and, in the non-relativistic limit.

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## REFERENCES

1. C. Doran, A. Lasenby and S. Gull, “States and operators in the space-time algebra,” *Found. Phys.* **23**, 1239-1264 (1993).
2. F. Gross, *Relativistic Quantum Mechanics and Field Theory* (Wiley Interscience, New York, 1993).
3. A. Gsponer, “On the “equivalence” of the Maxwell and Dirac equations,” *Int. J. Theor. Phys.* **41**, 689-694 (2002).
4. S. Gull, A. Lasenby and C. Doran, “Imaginary numbers are not real: The geometric algebra of spacetime,” *Found. Phys.* **23**, 1175-1201 (1993).
5. S. Gull, A. Lasenby and C. Doran, “Electron paths, tunnelling and diffraction in the spacetime algebra,” *Found. Phys.* **23**, 1329-1356 (1993).
6. D. Hestenes, *Space-Time Algebra* (Gordon & Breach, New York, 1966).
7. D. Hestenes, “Local observables in quantum theory,” *J. Math. Phys. (N.Y.)* **14**, 893-905 (1973).
8. D. Hestenes, “Observables, operators and complex numbers in the Dirac theory,” *J. Math. Phys. (N.Y.)* **16**, 556-572 (1975).
9. D. Hestenes, “Spacetime structure of weak and electromagnetic interactions,” *Found. Phys.* **12**, 153-168 (1982).
10. D. Hestenes, “Zitterbewegung modeling,” *Found. Phys.* **23**, 365-387 (1993).

11. N. Mukunda and R. Simon, “Quantum kinematic approach to the geometric phase I: General formalism,” *Ann. Phys. (N.Y.)* **228**, 205-268 (1993).
12. J.J. Sakurai, *Modern Quantum Mechanics* (Addison-Wesley, Reading, MA, 1994).
13. Z. Wang and B. Li, “Geometric phase in relativistic quantum mechanics,” *Phys. Rev. A* **60**, 4313-4317 (1999).